

Topological Field Configurations in the Presence of Isospin Chemical Potential

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We analyze the stability of different topological solutions in Quantum Field Theory when an isospin chemical potential μ is included. We work in the limit when temperature vanishes. We find that static vortex solutions in $2+1D$ do exist. However, the 't Hooft-Polyakov monopole in $3+1D$ is no longer stable, as soon as the chemical potential acquires a finite value. In the case of the Skyrmion, this topological solution still exists for finite μ , up to a certain critical value.

The possibility of a new mechanism leading to spontaneous symmetry breaking induced by the Bose-Einstein condensation, due to chemical potentials, has recently been explored in the frame of the electro-weak model [1]. There is a crucial difference with the standard symmetry breaking mechanism since now the number of Nambu-Goldstone bosons that appear is lesser than the number required by the Goldstone theorem [2].

In this letter, we explore the effect of the isospin chemical potential on some topological structures that appear in Quantum Field Theory (QFT). First, we will analyze the possibility of having vortex-type solutions in a $U(1)$ invariant theory in $2+1$ dimensions. Later, we address the question of the existence of 't Hooft-Polyakov monopole solutions in $3+1$ dimensions, finding that this kind of solutions are not longer topological stable, since the homotopy group becomes trivial for finite chemical potential values. Finally, we concentrate on Skyrmons, showing that, contrary to the monopole solutions, the chemical potential does not prevent their existence.

VORTEX IN $2+1D$

Let us consider a complex field ϕ in $2+1$ dimensions

$$\phi = \phi_1 + i\phi_2, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (1)$$

with a $U(1)$ -invariant lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial_\mu \phi - \frac{\lambda}{2} (\phi^* \phi - F^2)^2. \quad (2)$$

Following 't Hooft [4], the vacuum is defined through the manifold:

$$|\phi| = F. \quad (3)$$

Given a certain possible solution

$$|\vec{x}| \rightarrow \infty : \vec{\phi} = F \frac{\vec{x}}{|\vec{x}|}, \quad \phi = F e^{i\phi}, \quad (4)$$

the energy is given by

$$E = \int d^2x \left(\vec{\partial}\phi^* \vec{\partial}\phi + V(\phi, \phi^*) \right), \quad (5)$$

which diverges, since

$$|\vec{x}| \rightarrow \infty : (\partial_i \phi_j)^2 \rightarrow \frac{F^2}{|\vec{x}|^2}. \quad (6)$$

Then

$$\int d^2x \vec{\partial}\phi^* \vec{\partial}\phi = 2\pi \int d|\vec{x}| \frac{F^2}{|\vec{x}|^2} \sim \text{div log}. \quad (7)$$

The previous divergence can be avoided through the introduction of a covariant derivative.

$$\partial_\mu \rightarrow D_\mu \phi = (\partial_\mu - ieA_\mu) \phi. \quad (8)$$

The usual asymptotic finite energy solution, a vortex, is given by

$$\begin{aligned} \phi &= \Omega F, \quad \Omega = e^{i\varphi}, \\ \vec{A} &= -\frac{1}{ie} \Omega \vec{\partial} \Omega^{-1}, \\ A_i &= -\frac{1}{e} \epsilon_{ij} \frac{x_j}{r^2}. \end{aligned} \quad (9)$$

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To get a better understanding, we can analyze this solution in polar coordinates

$$\vec{A} = (A_r, A_\varphi) \rightarrow \left(0, \frac{1}{er}\right). \quad (10)$$

Using

$$\vec{A} = -\frac{1}{ie} e^{i\varphi} \vec{\partial} e^{-i\varphi} = \frac{1}{e} \vec{\varphi}, \quad (11)$$

the magnetic flux $\vec{B} = F_{12}$ inside a circle of area s is

$$\begin{aligned} \Phi &= \int_s \vec{B} \cdot d\vec{\sigma}, \\ &= \int_{c=\partial s} \vec{A} \cdot d\vec{l} = \int_{c=\partial s} \frac{1}{e} \vec{\nabla} \varphi \cdot d\vec{l}, \\ &= \frac{1}{e} \int_{c=\partial s} d\varphi = \frac{2\pi}{e}. \end{aligned} \quad (12)$$

This means that we have a quantized magnetic flux, in units of $g_m = 2\pi/e$.

The energy of this solution is

$$\begin{aligned} E &= \int d^2x \left[D_i \phi^\dagger D_i \phi + \frac{1}{2} F_{12}^2 + \frac{\lambda}{2} (\phi^\dagger \phi - F^2)^2 \right] \\ &= \int d^2x \left[(\partial_i \phi)^2 + e^2 \vec{A}^2 \phi^2 + \frac{1}{2} F_{12}^2 + \frac{\lambda}{2} (\phi^\dagger \phi - F^2)^2 \right]. \end{aligned}$$

Choosing the following parameter values, since we know that $[\lambda] \sim [e^2]$ y $[eF] \sim m$ from dimensional analysis:

$$\lambda = e^2, \quad m = m_\phi = m_A = \sqrt{2}eF, \quad (13)$$

we have

$$\begin{aligned} E &= \int d^2x \left[(\partial_i \phi \pm e\epsilon_{ij} A_j \phi)^2 \right. \\ &\quad \left. + \frac{1}{2} \left(F_{12} \pm \sqrt{\lambda} (\phi^2 - F^2) \right)^2 \pm eF^2 F_{12} \right], \\ &\geq eF^2 \left| \int d^2x F_{12} \right| = eF^2 \frac{2\pi}{e} = \frac{\pi m^2}{e^2}. \end{aligned} \quad (14)$$

Let us introduce our chemical potential in the model. Now, the lagrangian density becomes modified in the usual way [3] according to

$$\mathcal{L} = \mathcal{D}_\mu \phi^* \mathcal{D}_\mu \phi - \frac{\lambda}{2} (\phi^* \phi)^2, \quad (15)$$

with

$$\mathcal{D}_i = D_i, \quad \mathcal{D}_0 \phi = \partial_0 \phi - i\mu \phi, \quad \mathcal{D}_0 \phi^* = \partial_0 \phi^* + i\mu \phi^*, \quad (16)$$

choosing $A_0 = 0$, which is by far a non trivial gauge election, the lagrangian density is

$$\begin{aligned} \mathcal{L} &= D_\mu \phi^* D^\mu \phi - \frac{\lambda}{2} \left[(\phi^* \phi)^2 - 2 \frac{\mu^2}{\lambda} \phi^* \phi + \left(\frac{\mu^2}{\lambda} \right)^2 \right] \\ &\quad + \frac{\mu^4}{2\lambda}. \end{aligned} \quad (17)$$

Neglecting the last constant term, and noticing that the chemical potential plays the same role as the term F in (2), we have that, for the same election of (13), there is a nontrivial minimum for the energy:

$$E \geq 2\pi \left(\frac{\mu^2}{\lambda} \right). \quad (18)$$

It is interesting to remark that the chemical potential can be considered as a possible source of symmetry breaking.

'T HOOFT-POLYAKOV MONOPOLE SOLUTION (3+1D)

Let us consider in what follows the $SO(3)$ invariant lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi), \quad (19)$$

where we introduced the definitions

$$\begin{aligned} F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc} A_\mu^b A_\nu^c, \\ D_\mu \phi^a &= \partial_\mu \phi^a + e\epsilon^{abc} A_\mu^b \phi^c, \\ V(\phi) &= \frac{\lambda}{4} (\phi^a \phi^a - F^2)^2. \end{aligned} \quad (20)$$

The energy is given by

$$E = \int d^3x \left(\frac{1}{2} (\vec{D}\phi^a)^2 + \frac{\lambda}{4} (\phi^a \phi^a - F^2)^2 + \frac{1}{2} (\vec{B}^a)^2 \right). \quad (21)$$

Asymptotically, the monopole-type solution points radially. In fact, the following ansatz satisfies this condition.

$$\begin{aligned} \phi^a &= \frac{r^a}{er^2} H(eFr), \\ A_0^a &= 0, \\ A_i^a &= -\epsilon^{aij} \frac{r_j}{er^2} [1 - K(eFr)]. \end{aligned} \quad (22)$$

Notice that this solution is only invariant under the diagonal subgroup of $SO(3)_R \otimes SO(3)_G$. In the previous expressions, the functions H and K obey the boundary conditions

SKYRMIONS

$$\begin{aligned} K(\xi) &\rightarrow 1, \quad H(\xi) \rightarrow 0, \quad \text{when } \xi \rightarrow 0, \\ K(\xi) &\rightarrow 0, \quad \frac{H(\xi)}{\xi} \rightarrow 1, \quad \text{when } \xi \rightarrow \infty, \end{aligned} \quad (23)$$

with $\xi = eFr$.

With this solution, the energy is given by

$$\begin{aligned} E = \frac{4\pi F}{e} \int_0^\infty &\left[\xi^2 \left(\frac{dK}{d\xi} \right)^2 + K^2 H^2 + \frac{1}{2} \left(\xi \frac{dH}{d\xi} - H \right)^2 \right. \\ &\left. + \frac{1}{2} (K^2 - 1)^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \right]. \end{aligned} \quad (24)$$

According to the usual recipe, the lagrangian density when isospin chemical potential is introduced, becomes modified as

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) \\ & + \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2). \end{aligned} \quad (25)$$

The last two terms contribute to the energy density according to

$$\begin{aligned} \Delta\mathcal{E} = & V(\phi) - \frac{\mu^2}{2} (\phi_1^2 + \phi_2^2), \\ = & \frac{\lambda}{2} (\vec{\phi}^2 - \zeta^2)^2 + \frac{\mu^2}{2} \phi_3^2, \end{aligned} \quad (26)$$

with

$$\zeta^2 = \frac{F^2}{2} + \frac{\mu^2}{4\lambda}. \quad (27)$$

Note that the last term in (26) breaks explicitly the diagonal $SO(3)$ symmetry. In order to have a finite energy solution, we need

$$\phi_3(r \rightarrow \infty) = 0, \quad (\phi_1^2 + \phi_2^2)(r \rightarrow \infty) = \zeta^2, \quad (28)$$

i.e. the vacuum manifold reduces to S_1 . Since the homotopy group $\pi_2(S_1) = 0$, we conclude that there is no room for the existence of 't Hooft-Polyakov monopoles in the presence of a finite isospin chemical potential, which is equivalent to a charge asymmetric state of matter. It is possible, nevertheless, to have an accidental monopole solution when the original potential becomes

$$V(\phi) = \frac{\lambda}{2} (\vec{\phi}^2)^2 - \frac{m^2}{2} \phi_3^2, \quad (29)$$

when $\mu = m$. In this particular case the rotational field symmetry is restored.

Skyrme's original idea [5] was to consider the baryons as solitons in the non-linear sigma model. Later, this idea was fairly well supported by the predictions of static properties of baryons [6], being the baryonic number explained as a topological effect [7]. Extensions of this idea to the Hybrid models, the Cheshire Cat scenarios at finite temperature were also considered [8].

Due to the phenomenological success of the Skyrme model, it is natural to analyze the influence of the isospin chemical potential on the Skyrme solution. The lagrangian of the non-linear sigma model with the stabilizing skyrme term is given by

$$\begin{aligned} \mathcal{L} = & \frac{F_\pi^2}{16} Tr [\partial_\mu U \partial^\mu U^\dagger] \\ & + \frac{1}{32e^2} Tr [(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2. \end{aligned} \quad (30)$$

As the chemical potential plays the role of a temporal gauge field, it is natural the generalization of the covariant derivative:

$$\partial_\nu U \rightarrow D_\nu U = \partial_\nu U - i \frac{\mu}{2} [\sigma^3, U] g_{\nu 0}. \quad (31)$$

Defining $L_\nu \equiv (\partial_\nu U) U^\dagger$, the lagrangian density becomes

$$\begin{aligned} \mathcal{L}_\mu = & \frac{F_\pi^2}{16} Tr \left\{ \partial_\nu U \partial^\nu U^\dagger + \frac{\mu^2}{2} [\sigma_0 - U \sigma_3 U^\dagger \sigma_3] \right\} \\ & + \frac{1}{32e^2} Tr \left\{ [L_\rho, L_\nu]^2 - \frac{\mu^2}{2} ([\varrho, L_\nu][\varrho, L^\nu]) \right\}, \end{aligned} \quad (32)$$

where $\varrho = \sigma_3 - U \sigma^3 U^\dagger$.

For static solutions, we obtain

$$\begin{aligned} \mathcal{L}_\mu = & \mathcal{L}_{\mu=0} + \frac{F_\pi^2 \mu^2}{32} Tr [\sigma_0 - U \sigma_3 U^\dagger \sigma_3] \\ & + \frac{\mu^2}{64e^2} Tr [\varrho, L_\nu]^2. \end{aligned} \quad (33)$$

We parameterize the field U matrix as

$$U = \exp \left(\xi \frac{\vec{\sigma}}{2i} \cdot \hat{n} \right) = \cos \frac{\xi}{2} - i (\vec{\sigma} \cdot \hat{n}) \sin \frac{\xi}{2}. \quad (34)$$

The ansatz by Skyrme has an ‘‘Hedgehog’’ shape and the following boundary conditions have to be satisfied

$$\begin{aligned} \xi(\vec{r}) &= \xi(r), \quad \hat{n} = \hat{r}, \\ \xi(0) &= 2\pi, \quad \xi(\infty) = 0. \end{aligned} \quad (35)$$

where $\xi(r)$ can be solved variationally. It is possible to find an approximate analytical solution [10], in terms of dimensionless parameters

$$\xi = 2\pi \left[1 - \frac{r}{\sqrt{\lambda^2 + r^2}} \right]. \quad (36)$$

The optimal value for λ , from minimizing the mass of the Skyrmion is $\lambda = 1.453$.

The chemical potential dependent mass is given by

$$M_\mu = M_{\mu=0} - \frac{\mu^2}{4e^3 F_\pi} I_2 - \frac{\mu^2}{32e^3 F_\pi} I_4, \quad (37)$$

where the dimensionless integrals I_2 and I_4 are

$$\begin{aligned} I_2 &= \int d^3r Tr [\mathbb{1} - U \sigma_3 U^\dagger \sigma_3], \\ I_4 &= \int d^3r Tr [\varrho, L_\nu]^2. \end{aligned} \quad (38)$$

Notice that the corrections coming from the chemical potential terms have a negative sign. This means that eventually, for some critical value of μ_c , the Skyrmion mass should vanish. This picture is attractive, since it suggests that nucleons disappear for critical values of density.

As an approximation, we will estimate these integrals using the ansatz (36), and the value of $M_{\mu=0} = 36.5 \frac{F_\pi}{e}$ given by [7], obtaining

$$M_\mu = 36.5 \frac{F_\pi}{e} - 33.72 \frac{\mu^2}{e^3 F_\pi}. \quad (39)$$

This gives a critical value

$$\mu_c = 1.04 e F_\pi \approx 731 MeV. \quad (40)$$

This value is rather high. However, we have presented only a first rough discussion of this issue. Later we will present a more detailed analysis of Skyrmion stability. Our results implies that nucleons, according to the Skyrmion description will be stable in heavy ion collisions and other scenarios where we expect to achieve isospin chemical potential values of the order of the pion mass.

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